A20.1. Mathematical details of the Bohm hidden variable model.

The Bohm model [1] piggybacks on a conservation law in quantum mechanics, so we will describe that law first.

Unitary time translation and conservation of probability in QM.

If the Hamiltonian is hermitian, $H^\dagger = H$, then the time translation operator is unitary:

$$U(t) = e^{iHt}, U^\dagger(t) = e^{-iH^\dagger t} = e^{-iHt} \qquad (1)$$
$$\Rightarrow U^\dagger(t)U(t) = 1$$

Thus the norm of the state vector stays the same for all time:

$$\langle \Psi(\mathrm{t})|\Psi(\mathrm{t})\rangle = \langle \Psi(0)U^\dagger(t)|U(t)\Psi(0)\rangle = \langle \Psi(0)|\Psi(0)\rangle \qquad (2)$$

This has the consequence that if the state vector splits into several parts, separated from each other in space, then the sum of the norms squared of the separate parts equals 1.

Differential conservation law for the wave function.
No spin, non-relativistic.

We have a single particle with Schrodinger time evolution

$$\partial_t\psi = -iH\psi = (-i)\left[\frac{-\hbar^2}{2m}\nabla^2 + V(r)\right]\psi \qquad (3)$$
$$\partial_t\psi^* = +iH\psi^* = (+i)\left[\frac{-\hbar^2}{2m}\nabla^2 + V(r)\right]\psi^*$$

$$\partial_t(\rho) = \partial(\psi^*\psi) =$$
$$\frac{-i\hbar^2}{2m}[\psi\nabla^2\psi^* - \psi^*\nabla^2\psi] = \qquad (4)$$
$$\frac{-i\hbar^2}{2m}\nabla \bullet [\psi\nabla\psi^* - \psi^*\nabla\psi] = -\nabla \bullet \mathrm{J}$$
$$J = \frac{i\hbar^2}{2m}[\psi\nabla\psi^* - \psi^*\nabla\psi]$$

Thus

$$\partial_t(\rho) + \nabla \bullet \mathrm{J} = 0 \qquad (5)$$

This is the differential conservation law.

Why is it called a conservation law? Because it implies probability is conserved. The norm of the state $\int dx\ \psi(x,t)|x\rangle$ is $\int dx\ \ \rho(x,t)$ and, for sufficiently large volume Ω

$$\partial_t \int_\Omega\ dx\ \rho(x,t) = \int_\Omega\ dx\ \partial_t \rho(x,t) = \\ -\int_\Omega\ dx\ \nabla \bullet \mathrm{J} = -\int_S\ dA \bullet \mathrm{J} = 0 \tag{6}$$

where the last equality comes from J being 0 on the distant (three-dimensional) surface S. Thus the integral of the probability density over all space, equal to the norm squared, is constant in time. Hence the name 'conservation law.'

A similar equation applies to spinless, non-relativistic systems consisting of several 'particles' which obey the Schrodinger equation

$$i\ \hbar \partial_t \Psi(\mathrm{r}_1, \mathrm{r}_2, \dots) = \mathrm{H}\Psi(\mathrm{r}_1, \mathrm{r}_2, \dots) \\ H = \frac{-\hbar^2}{2m} \sum \nabla_k^2 + \sum_{j,k} w(j-k) + \sum_k V(k) \tag{7}$$

with *w* being the internal particle-particle potential energy and *V* the external potential energy. The conservation law then become

$$\partial_t\ \rho(\mathrm{r}_1, \mathrm{r}_2, \dots, \mathrm{t}) + \sum_k \nabla_k \bullet J_k = 0 \\ J_k(\mathrm{r}_1, \mathrm{r}_2, \dots, t) = \frac{i\hbar^2}{2m}[\psi \nabla_k \psi^* - \psi^* \nabla_k \psi] \\ \rho(\mathrm{r}_1, \mathrm{r}_2, \dots, \mathrm{t}) = \psi^*(\mathrm{r}_1, \mathrm{r}_2, \dots, \mathrm{t}) \psi(\mathrm{r}_1, \mathrm{r}_2, \dots, \mathrm{t}) \tag{8}$$

General attempt to find a conservation law.

$$\partial_t\ \psi = -iH\psi \\ (\partial_t\ \ \psi)^\dagger = i\psi^\dagger H \\ \partial_t(\psi^\dagger \psi) = i(H\psi)^\dagger \psi - i\psi^\dagger H \psi \tag{9}$$

If this is to give a conservation law, then the RHS must be proportional to a gradient. . Note that if J works, then $J + \nabla \times \mathrm{K}$ works also because $\nabla \bullet \nabla \times \mathrm{K} = 0$.

Note also that because $H \approx \sqrt{\nabla^2}$ for photons, there doesn't seem to be a conservation law for photons. Thus it is quite possible that the Bohm strategy cannot be made to work for photons.

The spin ½ Pauli equation and its conservation law.

The Pauli equation is half way to a relativistic theory, so it is worth doing here. The wave functions are column vectors of length 2.

$$H = mc^2 + \frac{(p - eA/c)^2}{2m} + e\phi - \frac{e\hbar}{2mc}\sigma \bullet \mathrm{B} \tag{10}$$
$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

It appears that the *J* in this case will be the same as in the no-spin case. One could also add a 'K_i' term such as $\psi^\dagger\ \sigma_i\ \psi$.

The Bohm model.
Single particle, no spin.

We are now ready to describe the Bohm model for a single particle. For each point x, we define a velicity, v, from

$$J = \rho v \tag{11}$$

This defines a **trajectory** which goes through point x.

Construction of the trajectory: We call the initial point on a particular trajectory as x_0. The next point, reached at time δt, is $x_1=x_0+v(x_0,t)\ \delta t$. We then calculate v_1 at x_1, t+ δt
and set $x_2=x_1+v_1\ \delta t$, etc.

This defines a continuous infinity of trajectories, one for each starting point x_0. But we could imagine using a very fine net of starting points x so there is instead a discrete set of trajectories.

Suppose we now do one run of whatever experiment we are considering. It is then assumed that, associated with the wave function on that run, there is an actual particle placed on one of the trajectories, with the particle having velocity v(x)=$J(x)/\ \rho(x)$. The hidden variables at a particular instant are then the position x and the velocity v. So this is essentially assuming the 'simplistic' view that there is a particle embedded in the wave function. Note that a 'particle' is arbitrarily placed on **just one** of the many possible trajectories. For a 'single-particle' wave function, one never places two 'particles' on different trajectories even though that is not prohibited by the mathematics.

There is no real reason to call this point a 'particle,' so we will call it the Bohm "system point."

Density of trajectories.

On each run of the experiment, the (conjectured) incoming system point will be on a definite trajectory, depending on its preparation history. The density of trajectories for the Bohm model, $\rho_B(x,t)$, is then **arbitrarily assumed** to be equal to

$$\rho_B(x,t) = \psi^*(x,t)\psi(x,t) \tag{12}$$

The meaning of $\rho_B(x,t)$ is that $d^3x\rho_B(x,t)$ is the probability of finding the system point on a trajectory in the element d^3x centered on *x*. But the Bohm density $\rho_B(x,t)$ is the same as the quantum mechanical density, $\rho_{QM}(x,t)$, which is the probability of finding 'the particle' at point *x*.

Probability.

Supppose we do a Stern-Gerlach experiment. At time t(0), before the magnet, the wave function is non-zero only in a region Ω_0. At time t(1), the wave function has split into two disjoint parts, non-zero only in the regions $\Omega(1)$ and $\Omega(2)$. The *quantum* probabilities for being on branch 1 or 2 are then resp.

$$\int_{\Omega(1)} d^3x\psi^*(x,t_2)\psi(x,t_2) = |a(1)|^2 \tag{13}$$
$$\int_{\Omega(2)} d^3x\psi^*(x,t_2)\psi(x,t_2) = |a(2)|^2$$

But because the Bohm probability density is the same as the quantum probability density, we will get that the *Bohm* probability of the trajectory entering branch 1 is $|a(1)|^2$ and the Bohm probability of the trajectory entering branch 2 is $|a(2)|^2$. That is, no matter how complicated the trajectories (and they are indeed complicated), just the right number of them will enter branch 1 so the probability law is satisfied. This happens because Bohm chose the trajectory density to be identical to the quantum density.

This ends our brief description of the Bohm model. See Ch. 20 for the weaknesses of the model.

References

[1] David Bohm, "A suggested interpretation of quantum theory in terms of "hidden variables," *Phys. Rev*. **85** 166,180 (1952).