A17.1 Derivation of classical mechanics from quantum mechanics.

We will show that, under certain assumptions, Newton's **F**=m**a** is a consequence of non-relativistic quantum mechanics.

N particles, classical.

We first consider the classical case of N particles, with the ith particle having coordinate x_i and mass m_i. The ith and jth particle have a potential energy $v(x_i-x_j)=v(i-j)$ between them, while the ith particle also travels under the effects of a potential energy $V(x_i)=V(i)$. The force is proportional to the gradient of the total potential energy, so Newton's law, $F_i=m_ia_i$ for the ith particle becomes

$$-\nabla_{i,k}V(i) - \nabla_{i,k}\sum_{j\neq i} v(i-j) = m_i \frac{d^2 x_{i,k}}{d^2 t}, \quad k = 1,2,3 \tag{1}$$

If we let the total mass be $M=\Sigma m_i$ and consider the motion of the center of mass coordinate, $X_{cm} = \Sigma m_i x_i / M$, then we obtain

$$M\frac{d^2 X_k}{d^2 t} = -\sum_j \nabla_{j,k} V(j), \quad k = 1,2,3 \tag{2}$$

where the gradients of the *v*(*i-j*) terms cancelled out.

N 'particles,' quantum mechanics.

We now consider the quantum mechanical case for N 'particles.' The equation for the wave function is

$$i\hbar\frac{\partial}{\partial t}\Psi(1,\dots,N,t) = H\Psi(1,\dots,N,t) \tag{3}$$
$$H = -\sum \frac{\hbar^2}{2m_i}\nabla_i^2 + \sum V(i) + \sum_i\sum_j v(i-j)$$

To obtain a match to the classical case, we consider the time evolution of the *expectation value* of the c.m. coordinate. We use the general rule

$$\frac{d}{dt}\langle\Psi(t)|O|\Psi(t)\rangle = \frac{-i}{\hbar}\langle\Psi(t)[H,O]|\Psi(t)\rangle \tag{4}$$

applied twice to the expectation value of $MX_{cm} = \sum m_i x_i$ to obtain

$$M\frac{d^2}{dt^2}\langle\Psi(t)|X_{k,cm}|\Psi(t)\rangle = -\sum_{j=1}^{N}\nabla_{j,k}\langle\Psi|V(j)|\Psi\rangle \tag{5}$$

This gives us a match between classical mechanics and quantum mechanics provided we can justify equating the quantum mechanical expectation values of the coordinates with the classical values.

Comparison of classical and quantum mechanics.

In the usual classical case, the wave function Ψ typically has its center of mass localized to a very small region, say on the order of 10^{-7} m or smaller, so the expectation value of the center of mass corresponds to a classical position.

But even if Ψ has a spread-out center of mass (that is, Ψ is non-zero for a relatively large range of the center of mass coordinate), we can fall back on the analysis of A14.2. There, the full wave function is broken up into a sum of wave functions, each of which is non-zero only for a very small range of the center of mass coordinate. And each of the terms in the sum evolves separately. Then we can apply the above analysis to each term in the sum separately.

Also, if the wave function evolves into a sum of isolated branches, the analysis applies to each branch separately. Thus classical mechanics, both statics (zero acceleration) and dynamics, follows from quantum mechanics.

This means classical mechanics, applicable to so many everyday situations and engineering problems, is another example of the accurate description of matter by quantum mechanics.

Note: Showing that quantum mechanics leads to classical mechanics is more difficult when one includes friction.