A16.1 Sketch of the Bell-Aspect reasoning and results.

The experiment [1,2]. **Entanglement**

An atom of calcium 40, originally in a spin 0 state, emits two photons (nearly) simultaneously, with one travelling to the left and the other to the right. The final state of the atom is also spin 0, so the two photons must have polarization states of spin 0, denoted by

$$|\psi\rangle = (|x\rangle_L |y\rangle_R - |y\rangle_R |x\rangle_L)/\sqrt{2} \tag{1}$$

Here the x and y refer to polarizations along the x and y axes resp. and the space-dependent part of the state has been left out. The states of these two ‘particles’ are said to be **entangled** because the state of one depends on the state of the other. The photon travelling to the left is split into its x′ and y′ components along axes rotated by an angle θ_L so that

$$\begin{aligned} |x\rangle_L &= \cos(\theta_L)|x'\rangle_L + \sin(\theta_L)|y'\rangle_L \\ |y\rangle_L &= \cos(\theta_L)|y'\rangle_L - \sin(\theta_L)|x'\rangle_L \end{aligned} \tag{2}$$

and similarly for the R polarization states,

$$\begin{aligned} |x\rangle_R &= \cos(\theta_R)|x'\rangle_R + \sin(\theta_R)|y'\rangle_R \\ |y\rangle_R &= \cos(\theta_R)|y'\rangle_R - \sin(\theta_R)|x'\rangle_R \end{aligned} \tag{3}$$

Detectors are put on each of the four primed paths to determine the polarization, $|x'\rangle$ or $|y'\rangle$ of each photon. See Fig. (1) in Ch. 16, with L1, L2, R1, R2 determining the polarization. The polarization states of the two photons are measured at different angles.

The probabilities predicted by quantum mechanics.

The state of Eq. (1) can be rewritten as

$$\begin{aligned} \sqrt{2}|\psi\rangle = {} & |x'\rangle_L |y'\rangle_R [\cos(\theta_L - \theta_R)] + \\ & |x'\rangle_L |x'\rangle_R [\sin(\theta_L - \theta_R)] + \\ & |y'\rangle_R |x'\rangle_R [-\cos(\theta_L - \theta_R)] + \\ & |y'\rangle_R |y'\rangle_L [\sin(\theta_L - \theta_R)] \end{aligned} \tag{4}$$

so, using the coefficient squared formula of Ch. 8, the probabilities of perceiving each of the four states predicted by quantum mechanics are

$$\begin{aligned} P_{x'y'} = P_{y'x'} &= \cos^2(\theta_L - \theta_R)/2 \\ P_{x'x'} = P_{y'y'} &= \sin^2(\theta_L - \theta_R)/2 \end{aligned} \tag{5}$$

General formulation of a hidden variable description [1,3].

The idea behind hidden variable proposals is that, to account for our perception of a single version of reality, and even more importantly to account for the probability law, it is assumed there is an objective (that is, single-version) reality underlying the wave functions (state vectors) of quantum mechanics. The state of that underlying reality is specified by hidden variables λ ('hidden' because they are not directly observable). Each incoming or initial state has an associated set of λs, with probability $\rho(\lambda)$ for set λ. The value of λ determines the outcome of any experiment on that state. The weighted average over the various λs is assumed to reproduce the quantum probabilities of the various outcomes.

This idea is put in equation form in the following way: We suppose that for given outcome *j*, the function *C*(*j*, λ) is 1 if the outcome associated with λ is *j* and 0 if the outcome is something else. Then if the hidden variable (*hv*) theory is valid, one must have, for outcome *j*,

$$P_{hv}(j) = \int d\lambda C(j,\lambda)\rho(\psi,\lambda) = P_{QM}(j) \quad (6)$$

So the trick in each particular case is to find a $C(j,\lambda)$ and a $\rho(\psi,\lambda)$ that satisfy this equation for every *j*—or else to show there can be no *C*, ρ that satisfy the equation for every *j*. Note that $C(j,\lambda)$ does not depend on the wave function ψ because λ is assumed to be sufficient to determine the outcome. Also $\rho(\psi,\lambda)$ does not depend on *j* because the preparation of the state (before the measurement is made) is assumed to be sufficient to determine the density. But in general, ρ must depend on the wave function.

> To see this, suppose we consider the two-slit experiment. There we see that the quantum mechanical outcome depends on $|\psi|^2$. But since *C* doesn't depend on the wave function, ρ must. It also seems quite natural for the density of hidden variables to depend on the particular shape and characteristics of the wave function. In addition, the density depends on ψ (in spades!) in the 'successful' Bohm hidden variable model.

Bell's inequality. Locality.

Eq. (6) in this two-photon case is

$$P_{x'y'} = \int d\lambda C_{x'y'}(\theta_L,\theta_R,\lambda)\rho(\lambda) = \cos^2(\theta_L-\theta_R)/2 \quad (7)$$

with similar equations for the other three Ps. Bell's seemingly reasonable **locality** assumption was that the result for x'_L, y'_L could not depend on the orientation, θ_R, of the polarization detector at the *distant* R location. And similarly for x'_R, y'_R and θ_L. Thus one has

$$C_{x'y'}(\theta_L,\theta_R,\lambda) = A_{x'_L}(\theta_L,\lambda)B_{y'_R}(\theta_R,\lambda) \quad (8)$$

with *A* and *B* equal to 1 or 0. Bell then showed that if the 'local' *C* of Eq. (8) is put into Eq. (7), the LHS satisfies an inequality [1,3]. This inequality can be shown both theoretically and experimentally (Aspect, [1]) to lead to a contradiction with the RHS of the equation. Thus there is no solution to Eqs. (7,8); that is, there can be no *A*, *B*, ρ and no set of λs that satisfy the equations. (For the algebra, see [3], where the *A* and *B* can be +1 and – 1 instead of our 1 and 0.)

The results of the experiment.

The experimental results of the Aspect experiment agree completely with the results predicted by quantum mechanics. But they violate the Bell inequalities by 20 or more standard deviations. So basic quantum mechanics correctly explains the results while local hidden variables fail completely.

Solution to the hidden variable equation.

If one could show there was no solution *at all* to Eq. (7), then one would have a proof that there can be no hidden variable interpretation of quantum mechanics. But alas there is a solution. Draw a circle of radius 1. The hidden variable is then the angle ϕ measured from the x-axis. At $\phi = \theta_L$, draw a diameter. Then go another angle β along the circle and draw another diameter, with

$$\frac{\beta}{2\pi} = \cos^2(\theta_L, \theta_R)\,/2$$

This divides the circle into four parts. We then set

$$\begin{gathered} \rho(\phi) = 1/2\pi \\ C_{x\prime y\prime} = 1, \qquad \theta_L \le \phi \le \theta_L + \beta \\ C_{x\prime x\prime} = 1, \qquad \theta_L + \beta \le \phi \le \theta_L + \pi \\ C_{y\prime x\prime} = 1, \qquad \theta_L + \pi_L \le \phi \le \theta_L + \beta + \pi, \\ C_{y\prime y\prime} = 1, \qquad \theta_L + \beta + \pi \le \phi \le \theta_L + 2\pi, \qquad \mathrm{mod}(2\pi) \end{gathered}$$

And *C*…=0 otherwise. Then $\int_0^{2\pi} d\phi\; C_{...}(\phi)\,\rho(\phi)$ gives the correct probability for each of the four measurements.

Note on Bell's locality assumption.

One can show by several means, even using something as simple as the double-slit experiment, that the path of the alleged particulate photon (or any other 'particle') must depend on the wave function. And this is certainly true in the Bohm hidden variable model [4]. For the entangled particles of the Bell-Aspect experiment, this introduces a *correlation* (not an interaction) between the readings of the right detectors with those of the left detectors. This is the reason why Bell's locality assumption, although intuitively agreeing with our ideas of particles, cannot be used to rule out hidden variable models; it ignores the long-range quantum mechanical correlations induced by entanglement.

References.

[1] J. S. Bell, "On the Einstein Podolsky Rosen paradox", *Physics*, 1, 195 (1964).
[2] A. Aspect, P. Grangier, and G. Rogers, "Experimental Realization of Einstein-Podolsky-Rosen-Bohm Gedankenexperiment: A New Violation of Bell's

Inequalities*,” Phys. Rev. Lett*. 47, 460 (1981).
[3] David Bohm, “A suggested interpretation of quantum theory in terms of
“hidden variables*,*” *Phys. Rev*. **85** 166,180 (1952).
[4] J.F. Clauser, M.A. Horne, A. Shimony, R.A. Holt, "Proposed experiment to test local hidden-variable theories", *Phys. Rev. Lett.* **23** (15): 880–4 (1969).