# A12.3 The allowed values of spin.

The abstract discipline of group representation theory gives results that apply to actual 'particles.' It is startling to see how such a seemingly mundane observation—the outcome of experiments doesn't depend on the orientation of the apparatus—plus purely abstract mathematics, can lead to relevant, experimentally verifiable results such as the quantization of spin.

**The relevance of group representation theory.**

Because the **experimental results** don't depend on orientation, the **equations** must be invariant under rotations. That implies the solutions can be **classified according to representations** of the group (Ch. 12 and A12.2).

**The commutation relations.**

The group of interest is the set of all rotations, *O*(3), in three dimensional space. Any continuous group is completely specified by the commutation relations of the generators of infinitesimal transformations. The commutation relations for the generators, $J_x$, $J_y$, $J_z$ of the rotation group are

$$[J_x, J_y] = iJ_z$$
$$[J_y, J_z] = iJ_x \qquad (1)$$
$$[J_z, J_x] = iJ_y$$

One can show that the operator $J^2 = J_x^2 + J_x^2 + J_x^2$, the *total* angular momentum or spin, commutes with all three generators. This implies it will have a constant eigenvalue—the same for each vector in the representation space—in any irreducible representation of *O*(3).

**Determination of the allowed values of spin from mathematics alone.**

It is convenient to define

$$J_+ = J_x + iJ_y \qquad (2)$$
$$J_- = J_x - iJ_y = J_+^\dagger$$

with

$$[J_+, J_z] = -J_+$$
$$[J_-, J_z] = +J_- \qquad (3)$$
$$J^2 = J_+J_- + J_z^2 - J_z$$
$$J^2 = J_-J_+ + J_z^2 + J_z$$

We suppose there is a vector which can be chosen as an eigenvector of both $J_z$ and $J^2$.

$$J^2|s\rangle = \lambda_0|s\rangle \qquad (4)$$
$$J_z|s\rangle = s|s\rangle$$

Then the first of Eqs. (3) implies $J_zJ(+)|s\rangle=(s+1)J(+)|s\rangle$, so J(+) is a 'raising' operator for the eigenvectors of $J_z$. We now apply $J_+$ to $|s\rangle$ $m$ times to get some multiple of $|s+m\rangle$. Then from the fourth of Eqs. (3), we have

$$\begin{aligned} J_-J_+|s+m\rangle &= (J^2 - J_z^2 - J_z)|s+m\rangle \\ &= (\lambda_0 - (s+m)^2 - (s+m))|s+m\rangle \end{aligned} \tag{5}$$

But the eigenvalues of $J_-J_+ = J_+^\dagger J_+$ must be real and greater than or equal to 0, so

$$\begin{aligned} \lambda_0 - (s+m)^2 - (s+m) \geq 0 \\ \lambda_0 - (s+m)^2 + (s+m) \geq 0 \end{aligned} \tag{6}$$

where the second line comes from considering $J_+J_-$. Thus there must be both a maximum and a minimum eigenvalue. That is, m, and thus the spin, has a finite number of values (for each irreducible representation). To find the allowed values, we use the following reasoning.

(01).. $J^2=J(-)J(+)+J^2(z)+J(z)$
(02) $J^2=J(+)J(-)+J^2(z)-J(z)$
Apply (01) to $|max\rangle$ and (02) to $|min\rangle$ to get (with $J(+)|max\rangle=J(-)|min\rangle=0$)
(03) $J^2=J^2_z(max)+J_z(max)$
(04) $J^2=J^2_z(min) - J_z(min)$
Let $J_z(max)=J_z(min)+M$ and set (4)=(3).
(05) $(J_z(min)+M)^2+(J_z(min)+M)=J^2_z(min)-J_z(min)$
(06) $J^2_z(min)+2MJ_z(min)+M^2+J_z(min)+M= J^2_z(min)-J_z(min)$
(07) $M(M+1)=(-2)(M+1)J_z(min)$
(08) $J_z(min)=-M/2$ !!
(09) $J_z(max)=+M/2$ with M integer.
(10) $J^2=\ldots$ from (03) or (04).