A12.2 Basic elements of representation theory.

1. Invariance group of a linear operator.

As an illustration, consider the equation

$$\left(\frac{\partial}{\partial u}\frac{\partial}{\partial \bar{u}}+\frac{\partial}{\partial v}\frac{\partial}{\partial \bar{v}}\right)\psi(u,v) \equiv O\psi(u,v)=0 \qquad (1)$$

where *u*,*v* are complex variables, *O* stands for the linear, partial differential operator, and the bar denotes complex conjugation. The form of this operator is invariant under the set of all unitary transformations of the *u*,*v* variables

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}, \quad A^* = A^{-1} \qquad (2)$$

That is, one can show that

$$O(u',v') = \frac{\partial}{\partial u'}\frac{\partial}{\partial \bar{u}'}+\frac{\partial}{\partial v'}\frac{\partial}{\partial \bar{v}'} = \frac{\partial}{\partial u}\frac{\partial}{\partial \bar{u}}+\frac{\partial}{\partial v}\frac{\partial}{\partial \bar{v}} = O(u,v) \qquad (3)$$

If one unitary transformation is multiplied by another, the product is also a unitary transformation, so the set of all unitary 2x2 transformations with determinant 1, SU(2), forms (part of) the invariance group of the operator *O*.

2. Basis vectors.

We define the operator *U*(*A*) such that

$$U(A)f(u,v) = f(u',v') \qquad (4)$$

where u', v' are defined in Eq. (2). Then if *f* is a solution to Eq. (1), we have

$$\begin{aligned} U(A)[O(u,v)\psi(u,v)] &= 0 \\ &= O(u',v')\psi(u',v') \\ &= O(u,v)\psi(u',v') \end{aligned} \qquad (5)$$

That is, if the original function is a solution to the equation, so is the function with the variables transformed. But the set of all functions $\psi(u',v') = \psi(a_{11}u + a_{12}v, a_{21}u + a_{22}v)$, as A runs through SU(2), are not (in general) linearly independent. If we start with the solution u^2 for example, then as A runs through SU(2), there are only three independent functions— u^2, *uv*, and v^2. We can think of these three functions as forming the basis for a three dimensional vector space which is closed under the unitary transformations of Eq. (2).

It is useful to introduce an SU(2)-invariant scalar product in this vector space. We choose

$$\langle f|g\rangle = \frac{2}{\pi^2}\int d[Re(u)]\int d[Im(u)]\int d[Re(v)]\int d[Im(v)]\,\bar{f}g\; e^{-|u|^2-|v|^2} \quad (6)$$
$$= \frac{2}{\pi^2}\int_0^\infty |u|d|u|\int_0^{2\pi} d\theta\int_0^\infty |v|d|v|\int_0^{2\pi} d\varphi\,\bar{f}g\,\exp(-|u|^2-|v|^2)$$

We then represent the functional basis vectors by kets according to

$$cu = |1/2\rangle,\ \ \mathrm{c}v = |-1/2\rangle \quad (7)$$
$$\langle i|j\rangle = \delta_{ij}$$

with c chosen so the norm is 1. That is, kets stand for functions of the underlying variables *u* and *v*.

The three functions u^2, uv, v^2, or, in ket notation,

$$au^2 = |1\rangle,\qquad \mathrm{b}uv = |0\rangle,\qquad av^2 = |-1\rangle \quad (8)$$

where *a* and *b* are chosen so that

$$\langle i|j\rangle = \delta_{ij}$$

are also carried into linear combinations of each other by the transformations of Eq. (2). Remembering that the kets stand for function of *u*,*v*, we have, summing *j* from 1 to 3, and using Eq. (7),

$$U(A)|i\rangle = {}^{[3]}R_{ji}(A)|j\rangle \quad (9)$$

where the trailing superscript indicates the *R*s, quadratic in the a_{ij}, form a three dimensional **representation** of SU(2) in the sense that multiplication is preserved. That is, if transformation *A* is followed by transformation *B* and *BA*=*C*, then ${}^{[3]}R(B)\,{}^{[3]}R(A) = {}^{[3]}R(C)$, where the multiplications are matrix multiplications. Similarly the four functions u^3, u^2v, uv^2, v^3, suitably normalized, form the basis for a four dimensional representation of SU(2), and so on.

3. Generators of transformations.

The structure of continuous groups is almost completely determined by transformations very near the identity. To illustrate, we will use O(3), the group of all rotations in three dimensions. Consider small rotations, ε, about the z-axis,

$$x' = x\cos(\varepsilon) + y\sin(\varepsilon) \cong x + y\varepsilon$$
$$y' = y\cos(\varepsilon) - x\sin(\varepsilon) \cong y - x\varepsilon \quad (10)$$
$$z' = z$$

This transformation of variables has the following effect on an arbitrary function;

$$U_z(\varepsilon)f(x,y,z) \cong f(x + y\varepsilon, y - x\varepsilon, z)$$
$$\cong f(x,y,z) + i\varepsilon\left[ix\partial_y - iy\partial_x\right]f(x,y,z) \quad (11)$$

$$= f(x, y, z) + i\varepsilon L_z f(x, y, z)$$

where the linear, first order differential operator L_z is the Hermitian **generator** of an infinitesimal rotation about the z axis. We can also construct generators for rotations around the x and y axes, with the results

$$L_x = i[y\partial_z - z\partial_y], \qquad L_y = i[z\partial_x - x\partial_z], \qquad L_z = i[x\partial_y - y\partial_x] \tag{12}$$

We find that the commutator of any two of these yields a generator back;

$$[L_x, L_y] = iL_z, \qquad [L_y, L_z] = iL_x, \qquad [L_z, L_x] = iL_y \tag{13}$$

More generally, every continuous group yields a finite number of linearly independent generators of infinitesimal transformations, and the commutator of any two of them is a linear combination of the generators. The commutation relations essentially define the group.

It is interesting that the three generators of SU(2),

$$\begin{aligned} S_x &= (1/2)(u\partial_v + v\partial_u) + h.a. \\ S_y &= (i/2)(v\partial_u - u\partial_v) + h.a. \\ S_z &= (1/2)\,(u\partial_u - v\partial_v) + h.a. \end{aligned} \tag{14}$$

obey the same commutation relations as the *L*s of Eq. (12);

$$[S_x, S_y] = iS_z, \qquad [S_y, S_z] = iS_x, \qquad [S_z, S_x] = iS_y \tag{15}$$

Therefore the two groups, O(3) and SU(2), must be essentially identical (actually there is a 2 to 1 map of SU(2) onto O(3)). It is also of interest that the set of all 2x2 complex matrices of determinant 1, SL(2), is homomorphic to the homogeneous Lorentz group of relativistic transformations.

4. Matrix representatives of linear operators.

One can obtain a matrix representation of a linear operator, such as a generator, by generalizing the idea behind Eq. (9). To verify that it is a faithful representation, we must show that the proper multiplication rule holds. If $R_{ij}(A)$ represents the matrix elements of the representation of A, then we have, with a sum over repeated indices,

$$U(A)|i\rangle = |j\rangle R_{ji}(A) \tag{16}$$

$$R_{ji}(A) = \langle j|U(A)|i\rangle \tag{17}$$

$$U(B)U(A)|i\rangle = U(B)|\mathrm{j}\rangle R_{ji}(A) = |k\rangle R_{kj}(B)R_{ji}(A) \tag{18}$$

But $U(BA)|i\rangle = |\mathrm{k}\rangle R_{ki}(BA)$ and so the proper matrix multiplication rule, $R(BA) = R(B)R(A)$, holds.

As an example of converting an operator to matrix form, consider the linear operator S_x of Eq. (14) and use the two dimensional basis of Eq. (7). We see that

$$S_x|1/2\rangle = (1/2)v = (1/2)|-1/2\rangle = R_{21}|-1/2\rangle$$
$$S_x|-1/2\rangle = (1/2)u = (1/2)|1/2\rangle = R_{12}|1/2\rangle \tag{19}$$

and so the two dimensional matrix representative of S_x is

$$S_x = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{20}$$

5. Irreducible representations.

The set of five functions, u,v,u^2,uv,v^2, are mapped into linear combinations of each other by a transformation from SU(2), so they form the basis for a representation of that group. But u and v alone also form the basis for a two dimensional representation, and u^2,uv,v^2 separately form the basis for a three dimensional representation. Thus the five dimensional representation is reducible. But the two dimensional and three dimensional representations cannot be further reduced. This idea of reducible and irreducible representations generalizes to representations of all groups.

6. Invariants.

The operator $L_x^2 + L_y^2 + L_z^2 = L^2$ from O(3) commutes with all three Ls. This implies it is invariant under all rotations. All the basis vectors in a given irreducible representation are eigenvectors of L^2, with the same eigenvalue, so L^2 is a multiple of the identity in each irreducible representation. Basis vectors for different irreducible representations have in general different values for the eigenvalue.

This idea also generalizes. For each group, there will be polynomials in the generators which commute with all the generators and are therefore invariant under transformation from the group. Basis vectors for each irreducible representation will be eigenvectors of each invariant, and each different irreducible representation will have a different set of eigenvalues. Basis vectors within an irreducible representation are usually chosen to be eigenvectors of one or more of the generators. For example, basis vectors for SU(2) are usually taken to be eigenvectors of s_z. If the representation is of dimension n+1, the n+1 eigenvalues are $-n/2, -n/2+1, \ldots, +n/2$.

For the inhomogeneous Lorentz group, there are two invariants, one corresponding to mass and one to spin so representations are labeled by m and S. The vectors within a representation are labeled by energy, momentum, and z component of spin. The true internal symmetry group is not currently known, so we don't know the invariant operators. The charges, which are the eigenvalues of the diagonal generators, label basis vectors within a representation.

7. The permutation group. Antisymmetry.

There is one other group whose representations are important in quantum mechanics, the permutation group. This group occurs when one has a linear operator in many variables, with the operator being invariant under permutations of variables. For example, if the linear operator is

$$\mathcal{O}(1,2) = \frac{\partial^2}{\partial^2 x_1} + \frac{\partial^2}{\partial^2 x_2} + V(|x_1 - x_2|)$$

and if you exchange variables 1 and 2, you get the same operator back;

$$\mathcal{O}(2,1) = \frac{\partial^2}{\partial^2 x_2} + \frac{\partial^2}{\partial^2 x_1} + V(|x_2 - x_1|) = \mathcal{O}(1,2)$$

As another example, consider the operator

$$\mathcal{O}(1,2,\dots,N) = \sum_{i=1}^{N} \partial^2/\,\partial^2 x_i + \sum_{j\neq \mathrm{i}} V(x_i, x_j)$$

Then it is easy to show that $\mathcal{O}(2,1,3,\dots,N) = \mathcal{O}(1,2,3,\dots,N)$, and more generally that $\mathcal{O}$ is invariant (does not change) under any permutation of the variables. The set of all such permutations forms a group, $\mathcal{P}(N)$, with *N*! elements. There are many representations of $\mathcal{P}(N)$, of varying dimensions and there will in general be solutions of $\mathcal{O}\Psi = 0$ corresponding to each representation. One can then catalog solutions according to which representations they belong to.

Symmetric and antisymmetric representations.

There are two representations of $\mathcal{P}(N)$ that are of particular interest in physics. The first is the symmetric representation, in which the function does not change under any permutation *p*. If *p* exchanges variables 1 and 2, for example, then one has, for a function belonging to the symmetric representation,

$$p(1 \leftrightarrow 2)\Psi_S(1,2,\dots,\mathrm{N}) = \Psi_S(2,1,\dots,\mathrm{N}) = \Psi_S(1,2,\dots,\mathrm{N})$$

All bosons—particles with integer spin (0,1,2,…)—belong to the one-dimensional symmetric representation.

The second representation of interest is the one-dimensional antisymmetric representation in which any exchange just gives a minus sign. For example

$$p(1 \leftrightarrow 2)\Psi_A(1,2,\dots,\mathrm{N}) = \Psi_A(2,1,\dots,\mathrm{N}) = -\Psi_A(1,2,\dots,\mathrm{N})$$

All fermions—particles with odd half integer spin (1/2,3/2,…)—belong to the antisymmetric representation. This representation is of particular interest in Part IV.

It is not currently known why only symmetric and antisymmetric representations occur in quantum mechanics.