A12.1 Invariance of an operator.

Invariance of the form.

By using a simple example, we will illustrate the idea of invariance of an operator under a transformation of variables (see also A1.1). Suppose we have the operator

$$O(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial x} + a\frac{\partial}{\partial y}\frac{\partial}{\partial y} \qquad (1)$$

where *a* is a constant. We now change variables to

$$\begin{aligned} x' &= cx + sy \\ y' &= -sx + cy \\ c &= \cos(\theta),\ \ s = \sin(\theta) \end{aligned} \qquad (2)$$

Then from

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y'} = c\frac{\partial}{\partial x'} - s\frac{\partial}{\partial y'} \\ \frac{\partial}{\partial y} &= \frac{\partial x'}{\partial y}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial y}\frac{\partial}{\partial y'} = s\frac{\partial}{\partial x'} + c\frac{\partial}{\partial y'} \end{aligned}$$

we can show that

$$O(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial x} + a\frac{\partial}{\partial y}\frac{\partial}{\partial y} \qquad (3)$$

$$= \frac{\partial}{\partial x'}\frac{\partial}{\partial x'} + \frac{\partial}{\partial y'}\frac{\partial}{\partial y'} + (a-1)\left(s^2\frac{\partial}{\partial x'}\frac{\partial}{\partial x'} + c^2\frac{\partial}{\partial y'}\frac{\partial}{\partial y'} - 2sc\frac{\partial}{\partial x'}\frac{\partial}{\partial y'}\right)$$

Thus if *a*=1, we have

$$O(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\frac{\partial}{\partial y} = \frac{\partial}{\partial x'}\frac{\partial}{\partial x'} + \frac{\partial}{\partial y'}\frac{\partial}{\partial y'} = O(x',y') \qquad (4)$$

That is, the form of the operator is exactly the same in the unprimed and primed (transformed) variables. This is the definition of invariance of an operator under a transformation of variables. If *a* is different from 1, the operator is not invariant under rotations because its form has changed (the part multiplied by (a-1)).

Generation of new solutions from old.

An interesting consequence of invariance is that one can generate new solutions of the equation from old. Suppose we have

$$O(x,y)f(x,y) = 0 \qquad (5)$$

We now rotate the variables and put the rotated variables, x', y', in *f* in place of the original variables. We then see that $O(x, y)f(x', y') = O(x', y')f(x', y') = 0$ where the first equality comes from Eq. (4) and the second from Eq. (5) (with a renaming of the variables). Thus from the original solution *f*(*x*,*y*), we get another solution $f(x', y')$ for each value of θ in Eq. (2).

Each new solution can be written as the sum of just a few solutions.

There are an infinite number of solutions, one for each value of θ, but each member of the set as θ runs from 0 to 2π can usually be written as a linear combination of just a few solutions. These few solutions form the basis for a representation of the invariance group (see A12.2).