A1.1 The three basic mathematical principles of quantum mechanics.

These three principles—Hilbert space structure, linearity, and invariance—are shared by all the forms of quantum mechanics, both basic vanilla and non-vanilla (Ch. 1). They, along with probability (Ch. 8), are the starting point and the principle input for all interpretations.

Hilbert space structure and kets.

The states of matter are mathematically represented by **vectors** (which can be column vectors or functions of some set of variables). These are usually denoted in quantum mechanics by the "ket" symbol, $|\ldots\rangle$, with the … replaced by some symbol denoting the specific vector (A6.2). The set of all relevant vectors forms a vector space. If two vectors, $|i\rangle$ and $|j\rangle$ belong to the vector space, then the linear combination a $|i\rangle$+b $|j\rangle$ also belongs to the vector space, where a and b are complex numbers.

A Hilbert space also contains a scalar product. If $|i\rangle$ and $|j\rangle$ are two vectors that belong to the vector space, their scalar product is denoted by $\langle i|j\rangle$. The norm squared of a vector $|i\rangle$ is $\langle i|i\rangle$.

It is often assumed there is a complete set of orthonormal **basis vectors**, $|k\rangle$, with $\langle k|k'\rangle = \delta_{kk'}$ Any vector in the space, say $|\psi\rangle$, can be written as a sum of these vectors;

$$|\psi\rangle = a(k)|k\rangle \tag{1}$$

with a sum implied over the repeated index *k*, and $a(k) = \langle k|\psi\rangle$. The calculation of the results of any measurement must be independent of the basis used.

Linearity.

Linear operators *O* are defined in a Hilbert vector space. They obey

$$O(a\,|i\rangle+b\,|j\rangle)=aO|i\rangle+bO|j\rangle \tag{2}$$

where the result of the operator acting on $|i\rangle$ on the RHS is independent of a, b and $|j\rangle$. This simple-looking definition is extremely powerful in basic quantum mechanics, where all operators are assumed to be linear. Some of its consequences are given in Ch. 10 and 11, and A10.1.

Invariance of operators.

An operator is invariant under certain transformations if its form doesn't change. As an example, consider the operator

$$\mathcal{O}(x,y,z) = \partial_x^2 + \partial_y^2 + \partial_z^2 \tag{3}$$

We now rotate the axes to the new variables

$$\begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}z \\ y' &= a_{21}x + a_{22}y + a_{23}z \end{aligned} \tag{4}$$

$$z' = a_{31}x + a_{32}y + a_{33}z$$

with $A^T A = I$. We then find that the form of the operator in the primed variables is the same as the original form,

$$\mathcal{O}(x, y, z) = \partial_x^2 + \partial_y^2 + \partial_z^2 = \partial_{x'}^2 + \partial_{y'}^2 + \partial_{z'}^2 = \mathcal{O}(x', y', z') \tag{5}$$

When combined with linearity and group representation theory, the principle of invariance is very powerful. As we shall see in Ch. 12 and A12.1, invariance implies that the seemingly classical-world, particle-like properties of mass, energy, momentum, total and z component of spin (angular momentum), and charge, along with the appropriate conservation laws, are all actually properties of the state vectors of quantum mechanics (whether or not particles exist).